Mid-term Exam

Time:	14:00–17:00, September 23, 2022.	Course name:	*Algebraic Number Theory*
Degree:	BIII & MII.	Year:	Autumn Semester; 2022–2023.
Course instructor:	Ramdin Mawia.	Total Marks:	30.

Attempt all of the following problems.

Algebraic Number Theory

1. Let K be an algebraic extension of $\mathbb{Q}$ and let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. Show that any nonzero prime ideal of $\mathcal{O}_K$ is maximal. **5**

2. When do we say that a ring is a Dedekind domain? Let A be a Dedekind domain with a finite number of prime ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. **5**

   (a) Show that for each i, there is an $x_i \in \mathfrak{p}_i$ such that $x_i \notin \mathfrak{p}_i^2$ and $x_i \notin \mathfrak{p}_j$ for all $j \neq i$.

   (b) Show that x_i generates $\mathfrak{p}_i$, i.e., $\mathfrak{p}_i = \langle x_i \rangle$.

   (c) Conclude that A is a PID.

3. Define a discrete valuation ring (DVR). **5**

   (a) Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Suppose $\mathfrak{m}$ is generated by $\pi \in A$, such that π is not nilpotent. Show that

      i. $\cap_{i \geqslant 1} \mathfrak{m}^i = (0)$;

      ii. every $x \in A$ can be uniquely written as $x = \pi^n u$ for some unit u in A and some integer $n \geqslant 0$;

      iii. A is an integral domain, and hence a DVR.

   (b) Is it true that the power series ring $\mathbb{Q}[[X]]$ is a DVR?

   (c) Let A be a DVR and B be its integral closure in a finite separable extension of its field of fractions. Show that B is a PID. [*Hint*: Use problem no. 2 above.]

4. Let p be a prime, $\omega \in \mathbb{C}$ be a primitive p^r-th root of unity (with $r \geqslant 1$) and let $K = \mathbb{Q}[\omega]$ be of degree n over $\mathbb{Q}$. Show that **5**

   (a) $\mathrm{D}_{K/\mathbb{Q}}(1, \omega, \ldots, \omega^{n-1})$ divides a power of p.

   (b) $\mathrm{D}_{K/\mathbb{Q}}(1, \omega, \ldots, \omega^{n-1}) = \mathrm{D}_{K/\mathbb{Q}}(1, 1-\omega, \ldots, (1-\omega)^{n-1})$

   (c) $\prod_{j \in S}(1-\omega^j) = p$, where $S = \{j : 1 \leqslant j \leqslant p^r, p \text{ does not divide } j\}$.

   (d) Show that the ring of integers of K is $\mathbb{Z}[\omega]$. [*Hint*: Any element α in the ring of integers can be written as
   $$\alpha = \frac{1}{d}[a_1 + a_2(1-\omega) + \cdots + a_n(1-\omega)^{n-1}]$$
   where $d = \mathrm{D}_{K/\mathbb{Q}}(1, 1-\omega, \ldots, (1-\omega)^{n-1})$ and the a_i are integers.]

5. State true or false, with brief but complete justifications (**any five**): **10**

   (a) If $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are prime ideals in a Dedekind domain A, then
   $$\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m} + \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_m^{f_m} = \mathfrak{p}_1^{\min\{e_1, f_1\}} \cdots \mathfrak{p}_m^{\min\{e_m, f_m\}}$$
   for integers $e_i, f_j \geqslant 0$.

   (b) The polynomial ring in two variables $\mathbb{Q}[X, Y]$ is a Dedekind Domain.

   (c) The polynomial ring $\mathbb{Z}[X]$ is integrally closed (in its quotient field).

   (d) In a number field K, it is possible that $\mathcal{O}_K \cap \mathbb{Q}$ is a strictly larger set than $\mathbb{Z}$.

   (e) The ring of integers in $\mathbb{Q}[\sqrt{7}]$ is $\mathbb{Z}[\sqrt{7}]$.

   (f) A Dedekind domain which is a UFD is necessarily a PID.

   (g) Let K be a number field of degree n. If $\{\beta_1, \ldots, \beta_n\} \subset \mathcal{O}_K$ is a basis of K over F, then it is necessarily an integral basis for K, that is, a basis for the ring of integers $\mathcal{O}_K$ over $\mathbb{Z}$.

   (h) Let A be a subring of B such that B is integral over A. Then for any nonzero prime ideal $\mathfrak{P}$ of B, $A \cap \mathfrak{P}$ is a nonzero prime ideal of A.

   (i) The absolute discriminant of a number field can be equal to 2022.